Name: ....

Maths Class: ....

Year 12

Mathematics Extension 1

HSC Course

Assessment 1

December 2021

Time allowed: 75 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-11
45 Marks

SECTION I: (5 Marks) Multiple Choice

Answers to these multiple choice should be written in sheet provided in the answer booklet
All questions are worth 1 mark

Allow about 7 minutes for this section

1	NSW sells customised number plates to customers consisting of 2 letters followed by 3 numbers. James Bond wants a set which looks like this: JB 007 He has been told (correctly) by the issuing authority that he is assured of getting the "JB" but that his chance of getting the numerical part is one in: A. 10 B. 5! C 1000 D 676 000
2	$\frac{d}{dx}\left(e^{2x^3+1}\right) =$ A. e^{2x^3+1} B. $6\,e^{2x^3+1}$ C. $6xe^{2x^3+1}$ D. $6x^2e^{2x^3+1}$
3	For what values of x is the curve $y = x^3 - 3x^2$ both concave down and decreasing? A. $0 < x < 2$ B. $0 < x < 1$ C. $1 < x < 2$ D. $x > 2$
4	In how many ways can the letters of the word ALABAMA be arranged to form other 7-letter words? A. 4! B. 6! C. 7! D. $\frac{7!}{4!}$
5	Find $\frac{d}{dx}(log_2\sqrt{4-x^2})$ A. $\frac{-2x}{4-x^2}$ B. $\frac{-x}{4-x^2}$ C. $\frac{-2x}{ln2(4-x^2)}$ D. $\frac{-x}{ln2(4-x^2)}$

SECTION II

Allow about 65 minutes to complete this Section

START EACH QUESTION ON A NEW PAGE

<u>QUESTION 6</u>: (8 Marks)

Marks

(a) (i) Fully factorise the Polynomial $P(x) = 4x^3 - 12x^2 + 9x - 2$. **3**

(ii) Without the use of calculus, sketch $y = P(x)$ **2**

(b) An SRC is to be chosen. There are 9 candidates from Year 11 and 10 from Year 12. From this, 6 are to be chosen from Year 11 and 8 from year 12.

(i) How many different SRC Committees can be formed? (leave your answer in unsimplified form) **1**

(ii) Two Senior Prefects must come from the 8 Year 12 representatives, of which Bill and Jack are two. How many different Senior Prefects can be chosen if Jack refuses to hold one of these positions if Bill is elected? **2**

QUESTION 7: (7 Marks)

(a) A spherical balloon is expanding so that its volume is increasing at a rate of 50 mm^3 per second. **2**

At what rate is the radius expanding when its radius is 10 mm ?

(Give your answer to 2 dec. places)

The volume of a sphere is given by $\boldsymbol{V = \frac{4}{3}\pi r^3}$

(b) The Mass of an animal kept in a Zoo is modelled by

$$M = 20 - 18e^{kt}$$

where M is the mass in kilograms, t is the age of the animal in months, and k is a constant

(i) Show that the equation $\frac{dM}{dt} = k(M - 20)$ is a solution to the model above **1**

(ii) What is the weight of the animal at its birth? **1**

(iii) If the animal weighs 10 kg after 2 months, what is its weight after 12 months? **3**

QUESTION 8: (7 Marks)

(a) Find the term independent of x in the expansion of $(2x^2 + \frac{1}{x})^9$ **2**

(b) A moon breaks away from the surface of a planet, and its distance, x, in millions of kilometres, from the surface of the planet, is given by

$$x = log_e(1 + t), \text{ after } t \text{ years.}$$

The moon undergoes no resistance from any medium, including gravity

(i) Sketch the graph of distance against time, for $t > 0$ **1**

(ii) From the graph, describe the distance of the moon from the planet as $t \rightarrow \infty$ **1**

(iii) Find the initial velocity. **2**

(iv) The Terminal Velocity of an object is the limiting velocity as time grows. Find the Terminal Velocity of this moon. **1**

QUESTION 9: (8 Marks)

Given the curve $y = \frac{1}{x^2+1}$

(i) Find $\frac{dy}{dx}$ **1**

(ii) Prove that $\frac{d^2y}{dx^2} = \frac{6x^2-2}{(x^2+1)^3}$ **1**

(iii) Find the turning point on the graph and determine its nature **2**

(iv) Find the Point(s) of Inflexion. **2**

(v) Hence sketch the graph using all of the information above **2**

QUESTION 10: (7 Marks)

(a) 8 women and 8 men are to be seated around a circular table for dinner.

(i) In how many ways is this possible, if they can sit anywhere? **1**

(Leave your answer in unsimplified form)

(ii) To be sociable, a decision is made that the men and women should sit in alternate seats. **2**

In how many ways can this be done?

(Leave your answer in unsimplified form)

(b) The polynomial $P(x) = 4x^3 - 12x^2 + 5x + p$ has roots $\alpha, \beta, and\ \gamma$

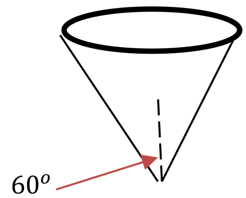
It is known that one root is the sum of the other 2 roots.

(i) Show that the product of the other 2 roots is -1. **3**

(ii) Hence find the value of p **1**

QUESTION 11: (8 Marks)

(a) Water is being poured into an inverted cone, of semi-vertical angle 60^o so that the water level (h) is rising constantly at a rate of $\frac{2}{\pi}$ mm per second.



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You may assume that the volume of water in the cylinder is $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the water level at any time.

(i) Show that $r = h\sqrt{3}$ **1**

(ii) Find the rate at which the volume is increasing when the water level is 20 mm. **3**

(iii) At what rate is the radius of the top of the water increasing at this time? **2**

(b) Using the expansion of $(x+1)^n$, show that **2**

$$\binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n-1} = 2^n - 2$$

YEAR 12 EXTENSION 1 MATHS

ASSESSMENT 1

Dec, 2021

SOLUTIONS

(1) C (2) D (3) $\frac{dy}{dx} = 3x^2 - 6x$. For decreasing

$\frac{dy}{dx} < 0 \Rightarrow 3x(x-2) < 0$

$\therefore 0 < x < 2$.

$\frac{d^2y}{dx^2} = 6x - 6$. For concave down $\frac{d^2y}{dx^2} < 0$

$\Rightarrow x < 1$.

$\therefore 0 < x < 1 \quad \therefore$ (B)

(4) $7!/4! \quad \therefore$ (D) (5) $\frac{d}{dx} \frac{1}{2}\left[\log_2(4-x^2)\right] = \frac{1}{2} \times \frac{-2x}{4-x^2} \cdot \frac{1}{\ln 2}$

$= \frac{-x}{\ln 2(4-x^2)}$.

$\therefore$ (D)

QUESTION 6:

(a) (i) $P(0) \neq 0$, $P(1) \neq 0$
$P(-1) \neq 0$
$P(2) = 0 \Rightarrow x-2$ is a factor

$$\begin{array}{r} 4x^2 - 4x + 1 \\ x-2 \overline{) 4x^3 - 12x^2 + 9x - 2} \\ 4x^3 - 8x^2 \\ \hline -4x^2 + 9x - 2 \\ -4x^2 + 8x \\ \hline x - 2 \end{array}$$

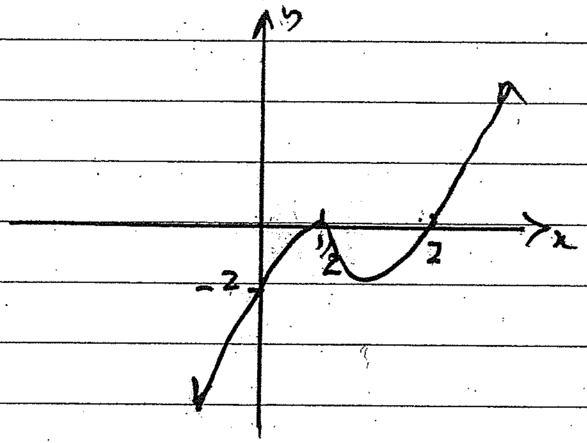
$\therefore$ $P(x) = (x-2)(2x-1)^2$

1 MARK for finding $x-2$

1 MARK for finding $4x^2-4x+1$

1 MARK for this form

(ii)



2 MARKS
1 for "bouncing" at $x = \frac{1}{2}$ and cutting at $x = 2$

1 for shaping up to the right and for $y = -2$

(b) (i) ${}^{9}C_{6} \times {}^{10}C_{8}$

1 MARK

(ii) With BILL elected, ${}^{6}C_{1} = 6$
OR/
with BILL out, ${}^{7}C_{2} = 21$

$\therefore$ no of ways $= 27$

1 MARK for considering both cases

1 MARK for 27.

QUESTION 7:

(a) $dV/dt = 50$	
$V = \frac{4}{3}\pi r^3 \Rightarrow dV/dr = 4\pi r^2$	1 MARK for dV/dr.
$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$	1 MARK. ignore units (accept $\frac{1}{8\pi}$)
$= \frac{1}{4\pi r^2} \times 50$	
$= \frac{50}{4\pi r^2}$	
At $r = 10$, $dr/dt = \frac{1}{8\pi}$	
≈ 0.04 mm/s	
(b) $m = 20 - 18e^{kt}$	
(i) $\frac{dm}{dt} = -18ke^{kt}$	1 MARK. Must be a feasible solution and not too many short cuts.
$= k(-18e^{kt} + 20) - 20k$	
$= km - 20k$	
$= k(m - 20)$	
(ii) At $t = 0$, $m = 2$ kg	1 MARK (ignore units)
(iii) At $t = 2$, $m = 10$	
$\therefore -10 = -18e^{2k}$	1 MARK for finding k.
$\therefore 2k = \ln(10/18)$	
$k = -0.294$	
At $t = 12$, $m = 20 - 18e^{12k}$	1 for $m = 20 - 18e^{12k}$
≈ 19.47 kg	1 for mass (ignore units)

QUESTION 8:

(a) Term is $^{9}C_3 \left(2x^2\right)^3 \left(\frac{1}{x}\right)^6$

$$= \frac{9\times8\times7}{1\times2\times3}(2)^3$$

$$= 84\times8$$

$$= 672$$

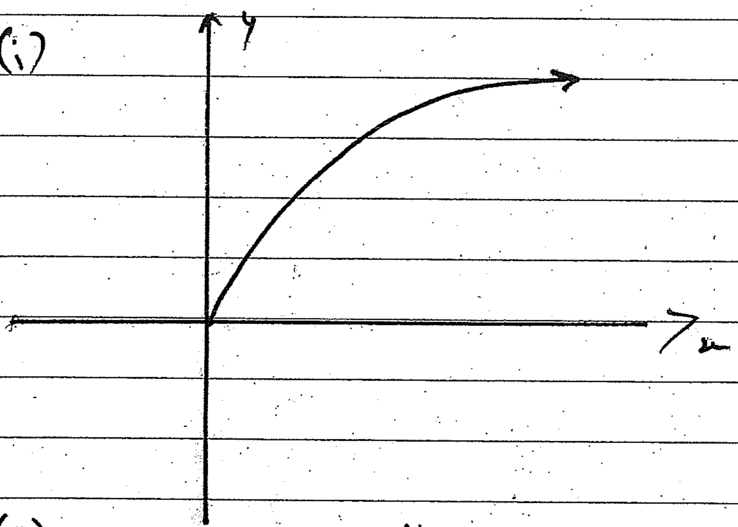
2 MARKS

[Accept $^{9}C_3 (2)^3$]

(lose 1 mark for each error)

(b) $x = \log_e(1+t)$

(i)



1 MARK

- Shape is important
- must start from (0,0)

(ii) As $t\to\infty$, the moon moves further away from the planet (more slowly as time goes on)

1 MARK

(no need to state the "more slowly" part)

(iii) At $t=0$, $\frac{dx}{dt} = \frac{1}{1+t}$

$= 1$ mkm/year

2 MARKS

1 for dx/dt

1 for 1mkm/year (ignore units)

(iv) As $t\to\infty$, $v\longrightarrow 0$

$\therefore$ terminal velocity is ZERO

1 MARK

QUESTION 9:

(i) $dy/dx = -(x^2+1)^{-2} \cdot 2x$

$= -2x/(x^2+1)^2$

1 MARK for either form

(ii) $$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2(-2) - (-2x)2.2x(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1) + 8x^2}{(x^2+1)^3}$$

$$= 6x^2 - 2/(x^2+1)^3$$

1 MARK.

(iii) At S.P.'s $dy/dx = 0$

$\therefore \begin{cases} x = 0 \\ y = 1 \\ y'' < 0 \Rightarrow \text{MAX. T.P.} \end{cases}$

2 MARKS

1 for finding (0, 1) (must have y = 1)

1 for testing y'' or similar

(iv) At I.P.'s $d^2y/dx^2 = 0$

$\therefore 6x^2 - 2 = 0$

$\therefore \begin{cases} x = 1/\sqrt{3} \\ y = 3/4 \end{cases}$ or $\begin{cases} x = -1/\sqrt{3} \\ y = 3/4 \end{cases}$

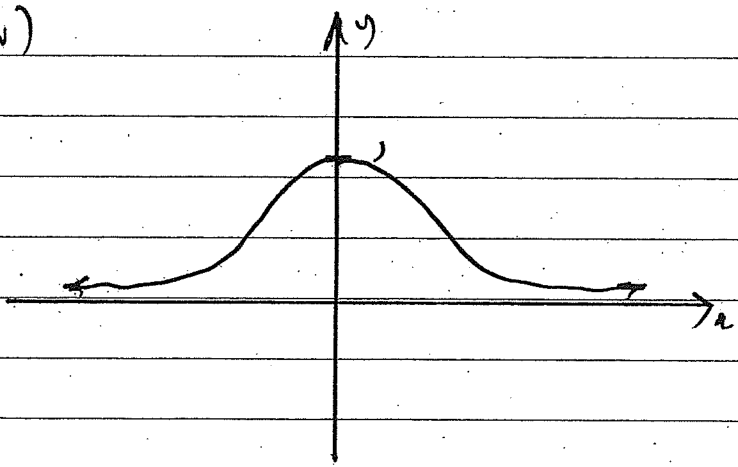
x	0	$1/\sqrt{3}$	1
y''	−	0	+

x	−1	$-1/\sqrt{3}$	0
y''	+	0	−

2 MARKS

- 2 for finding BOTH points and for testing (no need for y-values)
- 1 MARK for finding only one I.P. and testing
- 1 MARK for finding both I.P.'s but not testing

(v)



2 MARKS

Subtract 1 mark for missing each of these:

- MAX at (0, 1)
- shape
- limits as $x \to \pm\infty$

QUESTION 10:

(a) (i) $15!$ — 1 MARK.

(ii) Seating the men $= 7!$ ways — 1 for $7!$

Seating the women $= 8!$ ways — 1 for $8!$

$\therefore$ no of ways $= 7! \times 8!$ — [only 1 MARK for $7! \, 6!$]

(b) $P(x) = 4x^3 - 12x^2 + 5x + p$

Let the roots be α, β, γ

where $\alpha = \beta + \gamma$

(i) Sum $= \alpha + \beta + \gamma = 3$

$\therefore 2\alpha = 3$

$\alpha = 3/2$ — 1 for finding α

Product in pairs

$= \alpha\beta + \alpha\gamma + \beta\gamma = 5/4$ — 1 MARK.

$\therefore \alpha(\beta + \gamma) + \beta\gamma = 5/4$

$\therefore 9/4 + \beta\gamma = 5/4$

$\therefore \beta\gamma = -1$ — 1 MARK

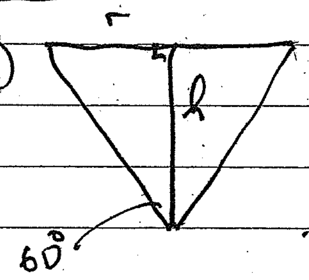
(ii) Product $= \alpha\beta\gamma = -p/4$

$= 3/2(-1) = -p/4$

$p = 6$ — 1 MARK.

QUESTION 11:

(a)(i)



$r/h = \tan 60°$

$= \sqrt{3}$

$\therefore r = h\sqrt{3}$. — 1 MARK (proof has to be substantial)

(ii) $V = 1/3 \pi r^2 h$

$= 1/3 \pi . 3h^2 h$

$= \pi h^3$.

$\therefore dV/dh = 3\pi h^2$ — 1 MARK for dV/dh.

and $dh/dt = 2/\pi$

$dV/dt = dV/dh \times dh/dt$	
$= 3\pi h^2 \times 2/\pi$	
$= 6h^2$	1 MARK for $dV/dt = 6h^2$
At $h = 20$, $dV/dt = 2400\ mm^3/s$	1 MARK (ignore units)
(iii) EITHER $dr/dt = dr/dh \times dh/dt$	2 MARKS
$= \sqrt{3} \cdot 2/\pi$	1 MARK for dr/dh
$= 2\sqrt{3}/\pi$	1 for answer.
OR	OR
$V = \frac{1}{3}\pi r^2 (r/\sqrt{3})$	
$= \pi r^3 / 3\sqrt{3}$	2 MARKS
$dV/dr = \pi r^2/\sqrt{3}$	1 MARK for dV/dr
$\therefore dr/dV = \sqrt{3}/\pi r^2$	
$dr/dt = dr/dV \times dV/dt$	
$= \frac{\sqrt{3}}{\pi r^2} \times 2400$	
At $r = 20\sqrt{3}$, $dr/dt = \frac{2400\sqrt{3}}{1200\pi}$	1 MARK.
$= 2\sqrt{3}/\pi$.	

(b) $(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \dots + \binom{n}{n-1}x + \binom{n}{n}$ [no MARKS]

Let $x = 1$

$\therefore 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n}$ — 1 MARK for letting $x = 1$ and getting this

$\therefore 2^n = 2 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1}$ — 1 for realising $\binom{n}{0} = \binom{n}{n} = 1$

$\therefore \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} = 2^n - 2$

2021 Extension 1 Mathematics Assessment Task 1 for the 2022 HSC
MARKERS COMMENTS

Question 6

a) i) students had to use the factor theorem or similar, to find the $(x-2)$ as one of the factors, and then proceed with long division to complete routine exercises. Several students attempted to factorise 'off the top of their head' completely missing the point of the question. Polynomial techniques are important and fairly easy extension 1 concepts that need to be mastered by all students attempting the course. Algebraic mistakes in the long division deprived some students of easy marks. It is straightforward easy to check your answer in this question and students are encouraged to do so always, particularly when the answer is subsequently needed for sketching the second part

ii) despite being told not to do so, several students wasted time with calculus when sketching. When a polynomial is fully factorised sketching is a basic exercise with only intercepts needing to be labelled (sub in $x = 0$ for y-intercept) and turning/inflexion points sketched sensibly in between. GRAPHING using a variety of techniques is an important technique that will be tested both at STHS and the HSC.

b) Students struggled with part ii). Many did not interpret the language correctly, the combinations were only from year 12, so the multiplication from part i) was not relevant. Students also need to type in their answer into a calculator to gauge the feasibility of the number. This will help them check their answer. Given that only 2 students were being picked from 8, the answer is likely to be in the tens, rather than the thousands.

As a strategy, students must decide whether the question is a permutation or a combination. With a combination, the ${}^nC_r$ formula is by far the most reliable approach.

Question 7

a) Generally well done, although a few students need to realise that a rate of change needs to be a derivative with respect to time. Some silly errors with reciprocating the wrong derivative.
b) Also quite well done. A few too many unnecessarily lost marks in i). You needed some kind of justification to replace $-18e^{kt}$ with $M - 20$, show the marker that you have not fudged it!
In ii), a few students just wrote 20kg for the weight at birth, assuming that this was the initial value without substituting $t = 0$.
In iii), a few students need to be careful of making up new laws of logarithms when solving equations. Also take great care with calculator work!

Question 8

(b) (i) This was not well done:

- The axes needed to be time on the horizontal and distance on the vertical.
- A logarithmic graph increases but increasingly slowly. How to shift it sideways (because of the $log_e(1+t)$) should be well known
- Time cannot be negative.
- A minor point, which was not marked down was that the question said t>0 which means there was an open circle about the origin.

(b) (iii) Differentiation of the logarithmic function to get velocity was, in many instances, very poor.

(b) (iv) Having an idea about limits when t $t \rightarrow \infty$ is a vital part of the calculus of this course, which, despite all the words, is all that this question was about.

Question 9 **Well where to start……**

i. Ok just take care
ii. Very poor overall – students did not recognise to either use quotient OR product rule – others made mistakes in their algebra.
iii. 1- a fraction can equal zero ONLY when the numerator can = 0.
2- you needed a correct y value which MUST be found from the original y = equation.
3- YOU MUST TEST CORRECTLY and state if it is a MAX or a MIN TP the test should be LABELLED correctly
iv. VERY POOR again see above points but with the second derivative. YOU NEEDED BOTH X values WITH the correct Y value AND TEST to get both marks
v. The graph – check that what you draw matches what you found – WHEN IN DOUBT PLOT A FEW POINTS especially as $x \to \infty$ NOTING the graph WAS EVEN would have helped

Question 10

a) (ii) Quite a few errors here. A common error was 8! x 7! x 2. Here, there is no need to multiply by 2 to account for seating a man or a woman first. Other students answered incorrectly with 8! x 8! . Remember to seat one person first, then account for everyone else.
b) (i). Many students struggled with this question. Common errors included:
- Incorrectly remembering the sums and products of roots formulas.
- Many students experiencing difficulties making an appropriate substitution to prove the product is equal to -1.
- Many students struggling to get $\alpha = \frac{3}{2}$

(ii) Here, students struggled to get the correct value for p if they had made errors in the previous part.

Question 11

a) (i) Drawing a quick diagram did wonders to gain this easy mark.
(ii) Many tried to differentiate V with both the variables r and h involved. To differentiate V with respect to h, it needs to be in terms of h first, so you need to sub $r = h\sqrt{3}$ into V. Some tried to differentiate V with respect to r for $\frac{dV}{dH}$.
(ii) Few recognised that they could quickly find $\frac{dr}{dt} = \frac{dr}{dh} \times \frac{dh}{dt}$ by differentiating r with respect to h from $r = h\sqrt{3}$. Some successfully differentiated $\frac{dV}{dr}$ but did not find the reciprocal of this to then find $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$.
b) Be very clear for a show question. Clearly state that $\binom{n}{0} = \binom{n}{n} = 1$.
Write the expansion of $(x+1)^n$ first, notice that the x's have disappeared in what you need to show and 2^n is involved on LHS so this is the hint to substitute x = 1.
Some differentiated both sides w.r.t x… this would have been suitable if the LHS term involved $n(x+1)^{n-1}$ before substituting a value.